Expansions for determinants and for characteristic polynomials of stochastic matrices[1]

B.Delyon
LIDS, MIT, Cambridge, MA, USA.

Abstract: An expansion of the determinant of any matrix in terms of row sums and off-diagonal entries is given and used to obtain expressions for the coefficients of the characteristic polynomial of stochastic matrices.

Key words: Deteminants, eigenvalues, stochastic matrices.

AMS subject classification: 15A15, 15A18, 15A51.

Abreviated title: Epansions for determinants.

Report Documentation Page

Form Approved
OMB No. 0704-0188

Public reporting burden for the collection of information is estimated to average 1 hour per response, including the time for reviewing instructions, searching existing data sources, gathering and maintaining the data needed, and completing and reviewing the collection of information. Send comments regarding this burden estimate or any other aspect of this collection of information, including suggestions for reducing this burden, to Washington Headquarters Services, Directorate for Information Operations and Reports, 1215 Jefferson Davis Highway, Suite 1204, Arlington VA 22202-4302. Respondents should be aware that notwithstanding any other provision of law, no person shall be subject to a penalty for failing to comply with a collection of information if it does not display a currently valid OMB control number.

1. REPORT DATE **AUG 1988**	2. REPORT TYPE	3. DATES COVERED **00-08-1988 to 00-08-1988**

4. TITLE AND SUBTITLE	5a. CONTRACT NUMBER
Expansions for determinants and for characteristics polynomials of stochastic matrices	5b. GRANT NUMBER
	5c. PROGRAM ELEMENT NUMBER

6. AUTHOR(S)	5d. PROJECT NUMBER
	5e. TASK NUMBER
	5f. WORK UNIT NUMBER

7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)	8. PERFORMING ORGANIZATION REPORT NUMBER
Massachusetts Institute of Technology,Laboratory for Information and Decision Systems,77 Massachusetts Avenue,Cambridge,MA,02139-4307	

9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)	10. SPONSOR/MONITOR'S ACRONYM(S)
	11. SPONSOR/MONITOR'S REPORT NUMBER(S)

12. DISTRIBUTION/AVAILABILITY STATEMENT
Approved for public release; distribution unlimited

13. SUPPLEMENTARY NOTES

14. ABSTRACT

15. SUBJECT TERMS

16. SECURITY CLASSIFICATION OF:			17. LIMITATION OF ABSTRACT	18. NUMBER OF PAGES	19a. NAME OF RESPONSIBLE PERSON
a. REPORT **unclassified**	b. ABSTRACT **unclassified**	c. THIS PAGE **unclassified**		**8**	

Standard Form 298 (Rev. 8-98)
Prescribed by ANSI Std Z39-18

We found in [1,page 208 last paragraph] a quite interesting result (stated here in theorem2) which is actually neither clearly stated nor proved. This result provides an expression for each coefficient of the polynomial det(P-I-λI) (P is a stochastic matrix) involving sums of products of p_{ij}'s (without any change in sign, as for the usual expansion which is a sum of minors). The aim of this paper is to prove it as a consequence of a particular expansion of determinants which is given in theorem1; this expansion is a sum of products of off-diagonal entries and negated row sums of the matrix (with no sign added).

The main tool used here is the W-graphs introduced in [1].

<u>Definition</u> [1]: *Let L be a finite set and let a subset W be selected in L. A graph on L is called a W-graph is it satisfies the following conditions:*

(1) every point $m \in L \backslash W$ is the initial point of exactly one arrow, and any arrow has its initial point in $L \backslash W$.

(2) there are no closed cycles in the graph.

Note that (2) may be replaced by

(2') every point $m \in L \backslash W$ is the initial point of a sequence of arrows leading to some point $n \in W$.

These W-graphs may be seen as disjoint unions of directed trees on L with roots in W.

<u>Notations</u>:

The set of W-graphs will be denoted by G(W).

Suppose that we are given a set of numbers p_{ij} ($i,j \in L$), then for any graph g on L we define the number $\pi(g)$ by:

$$(1) \qquad \pi(g) = \prod_{(m \to n) \in g} p_{m\,n}$$

$$\pi(\text{empty graph}) = 1.$$

For any subset W of L, we put;

$$(2) \qquad \sigma(W) = \sum_{g \in G(W)} \pi(g)$$

In particular, $\sigma(L) = 1$.

Theorem1: *Consider a nXn matrix $A=(a_{ij})$ with row sums $r_i = \sum_{i=1}^{n} a_{ij}$ and*

define
L={1,2,...,n+1} and

$$(3) \qquad \begin{aligned} p_{ij} &= a_{ij} & 1 \le i,j \le n \\ p_{i,n+1} &= -r_i & 1 \le i \le n. \end{aligned}$$

Then

$$(4) \qquad det(M) = (-1)^n \, \sigma(\{n+1\})$$

where σ is defined as above.

An easy consequence will be

theorem2: *Consider a nXn matrix $P=(p_{ij})$ with constant row sums $r_i=r$, then its characteristic polynomial has the form:*

$$(5) \qquad P(\lambda) = \sum_{i=1}^{n} \sigma_i \, (\lambda-r)^i$$

where

$$(6) \quad \sigma_i = \sum_{|W|=i} \sigma(W).$$

Remark: Note that this last result applies also to matrices M with different row sums, by considering the matrix M' obtained by adding to M one column containing the negated row sums and one zero row.

Proof of theorem1;

Consider the function Δ defined by

$$\Delta(A) = (-1)^n \, \sigma(\{n+1\}).$$

We have to show that $\Delta(A)=\det(A)$. This will be done by proving some properties of the function Δ.

Property1: If A has a zero column, then $\Delta(A)=0$.
       Denote by m the index of the zero column and put

$L = \{1, 2,..., n+1\}$, $W =\{n+1\}$, and $L' = L\backslash\{m\}$
G = set of W-graphs on L
G' = set of W-graphs on L'.

Note that, because of the zero-column property of A, any graph g of G satisfying $\pi(g)\neq 0$ will not have any arrow leading to m, so that g can be described as a graph g' of G' to which has been added an arrow leading from m to any other point $i\in L'$; we call this graph g(g', i). Because all these graphs are distinct (for distinct g' or i) we get:

$$\Delta(A)= \sum_{g\in G(W)} \pi(g) = \sum_{\substack{g'\in G'(W) \\ i\in L'}} \pi(g(g',i)) = \sum_{g'\in G'(W)} \pi(g') \sum_{i\in L'} p_{mi} = 0.$$

The last equality follows from the definition of the p_{ij}'s.
This ends The proof of property1.

<u>Property2</u>: The application Δ is invariant under permutation of indices of the matrix (that is by succesive permutation of rows and corresponding columns).

This property is obvious.

<u>Property3</u>: If A is a block-diagonal matrix $A=diag(A_1,.....,A_p)$, then $\Delta(A)=\Delta(A_1).....\Delta(A_p)$.

This has only to be proved for p=2 (for larger p, one can use a recursion). Denote by m the size of the matrix A_1 and let

$L' = \{1,2,...,m,n+1\}$, $L'' = \{m+1, ...,n, n+1\}$, $W = \{n+1\}$,
$G' = $ set of W-graphs on L',
$G'' = $ set of W-graphs on L''.

Then, by the same reasoning as in property1, any graph g of G such that $\pi(g)\neq0$ is constructed as the union of two graphs $g'\in G'$ and $g''\in G''$ with the point n+1 in common and we obtain

$$\Delta(A)= (-1)^n \sum_{g\in G(W)} \pi(g)= (-1)^m(-1)^{n-m} \sum_{\substack{g'\in G'(W) \\ g''\in G''(W)}} \pi(g')\pi(g'') = \Delta(A_1)$$

$\Delta(A_2)$.

<u>Property4</u>: If two matrices A_1 and A_2 are the same, except for one row, then $\Delta(A_1+A_2) = \Delta(A_1) + \Delta(A_2)$.

This comes from the fact that, for any W-graph g, this additivity property is satisfied by $\pi(g)$.

<u>End of the proof of theorem2</u>:

Property4 implies that if we want to prove that $\Delta(A) = det(A)$ for any matrix M, we have only to check this for matrices having one non-zero entry on each row.

By virtue of property1, this is true if there exists a zero-column. If there is not any zero-column, then there is exactly one non-zero entry in each row and in each column, and there exists a permutation of indices which transform A into $diag(A_1,...,A_p)$ for some matrices $A_1,...,A_p$, where the non-zero entries of A_i occur only

in positions immediately above the diagonal and in the lower-left corner, or A_i is a 1×1 matrix; i.e., A_i has the form (we give the picture for a 4×4 matrix):

$$\begin{pmatrix} 0\,a\,0\,0 \\ 0\,0\,b\,0 \\ 0\,0\,0\,c \\ d\,0\,0\,0 \end{pmatrix}$$

$a\neq0$, $b\neq0$, $c\neq0$, $d\neq0$.

If $p>1$, we get the result by induction. The problem is then reduced to the case $p=1$ and $n>1$, i.e. to the study of $\Delta(A)$ when the non-zero entries of A are above the diagonal and in the lower-left corner. In that case, the W-graphs g for which $\pi(g)\neq0$ are described by the following property:

for any $1\leq i\leq n$, the arrow starting from i leads to i+1 (1 if i=n) or n+1.

Such a graph is exactly determined by the arrows leading to n+1. For any $1\leq k\leq n$, there exist exactly $\binom{n}{k}$ W-graphs having k arrows leading to n+1, and all these graphs g satisfy $\pi(g) = (-1)^k\,\pi_0$, where π_0 is the product of the non-zero entries of A. Finally we obtain

$$\Delta(A) = (-1)^n \sum_{k=1}^{n} \binom{n}{k}(-1)^k\,\pi_0 = (-1)^{n+1}\,\pi_0 = \det(A).$$

This ends the proof of theorem1.

Proof of theorem2: Apply theorem1 to $P-\lambda I$. The row sums of this matrix are all equal to $r-\lambda$. Note that there exists a one-to-one map between the set of {n+1}-graphs on {1, 2,...., n+1} and the set of all W-graphs, W non-empty, on {1, 2,...., n}. This map associates with any {n+1}-graph g on the set {1, 2,...., n+1} the graph g' obtained by deleting the point n+1 and all arrows leading to it. Note that if g has k arrows leading to n+1, then, using (1) and (3), we obtain:

$$\pi(g) = (\lambda - r)^k \, \pi(g').$$

This equality, inserted in (2) and (4) leads to the result.

References

[1]   M.I.Friedlin, A.D.Wentzell,"Random Pertubations of Dynamical Systems", Springer-Verlag, 1984.